Lecture 7: Projective Modules and Projective Covers

Goal: Understand the definition and properties of projective modules, the role of projective covers in modular representation theory, their relationship with Brauer characters and decomposition matrices, and how to identify projective indecomposables.

1. Projective Modules

Definition 7.1. An A-module P is called *projective* if for every surjective A-module homomorphism $f : M \to N$ and every A-module homomorphism $g : P \to N$, there exists an A-module homomorphism $h : P \to M$ such that:

$$f \circ h = g.$$

Equivalently, P is projective if the functor $\mathrm{Hom}_A(P, -)$ is exact.

Examples:

- Free modules are projective.
- Direct summands of projective modules are projective.
- In semisimple rings, every module is projective.

2. Projective Indecomposable Modules (PIMs)

Definition 7.2. A projective module P is called *indecomposable* if it cannot be written as $P = P_1 \oplus P_2$ with $P_1, P_2 \neq 0$ projective modules.

Theorem 7.3. In modular representation theory (i.e., over $F[G]$ with $\mathrm{char}(F) = p \mid |G|$), the number of isomorphism classes of indecomposable projective modules equals the number of simple modules (i.e., the number of irreducible Brauer characters).

3. Projective Covers

Definition 7.4 (Projective Cover). Let S be a simple $F[G]$-module. A *projective cover* of S is a surjective homomorphism $\pi : P \to S$, where P is projective and $\ker(\pi) \subseteq \mathrm{Rad}(P)$ (the radical of P).

Theorem 7.5. Every simple module S over $F[G]$ has a unique (up to isomorphism) projective cover P_S, which is indecomposable and called the *projective indecomposable module* (PIM) associated to S.

4. Relation to Brauer Characters and Decomposition Matrix

Proposition 7.6. Let $\{\chi_i\}$ be the ordinary irreducible characters and $\{\varphi_j\}$ the irreducible Brauer characters. The decomposition matrix $D = (d_{ij})$ records how the reductions of χ_i decompose into φ_j.

Theorem 7.7. The character of the projective indecomposable module P_{φ_j} is given by the j-th column of the transpose of the decomposition matrix.

5. Structure of Projective Modules in $F[G]$

Definition 7.8 (Radical and Socle Series). Let M be a module. Its radical series is the descending sequence:

$$M \supseteq \mathrm{Rad}(M) \supseteq \mathrm{Rad}^2(M) \supseteq \cdots$$

Its *socle* is the sum of all simple submodules.

Theorem 7.9. Every projective indecomposable module has a simple top (the quotient $P/\mathrm{Rad}(P)$) and a simple socle. This matches the simple module it covers.

6. Example

Example 7.10 (PIMs for S_3 over $\mathbb{F}_3$)

Let $F = \mathbb{F}_3$. Then:

- S_3 has 2 irreducible Brauer characters φ_1, φ_2,
- The projective indecomposable modules P_1, P_2 correspond to these,
- One PIM has character values matching $\chi_1 + \chi_3$, the other $\chi_2 + \chi_3$.

These column sums appear in the decomposition matrix.

7. Counterexamples

Counterexample 7.11. Not every indecomposable module is projective. For instance, uniserial modules of length 3 with non-projective socle and head.

Counterexample 7.12. Projective modules are not necessarily irreducible, nor are they semisimple unless the group algebra is semisimple.

8. Summary

In this lecture we studied:

- The definition and properties of projective and projective indecomposable modules,
- The notion of projective covers and their uniqueness,
- How projective modules relate to decomposition matrices and Brauer characters,
- Examples illustrating their structure in group algebras.

Coming Up in Lecture 8: We'll introduce the use of the software *GAP* for computing character tables, decomposition matrices, and blocks, with real examples from A_5, S_4, and others.